Rayleigh-Taylor Instability of a Partially Ionized Plasma in a Porous Medium in Presence of a Variable Magnetic Field

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The Rayleigh-Taylor instability of a partially ionized plasma in a porous medium is considered in the presence of a variable magnetic field perpendicular to gravity. The cases of two uniform partially ionized plasmas separated by a horizontal boundary and exponentially varying density, viscosity, magnetic field and neutral particle number density are considered. In each case, the magnetic field succeeds in stabilizing waves in a certain wave-number range which were unstable in the absence of the magnetic field, whereas the system is found to be stable for potentially stable configuration/stable stratifications. The growth rates both increase (for certain wave numbers) and decrease (for different wave numbers) with the increase in kinematic viscosity, medium permeability and collisional frequency. The medium permeability and collisions do not have any qualitative effect on the nature of stability or instability.

1. Introduction

A comprehensive account of the Rayleigh-Taylor instability under various assumptions of hydromagnetics has been given by Chandrasekhar [1]. More often than not, a partially ionized plasma represents a state which exists in the universe, and there are several situations in which the interaction between ionized and neutral gas components becomes important in cosmic physics. Strömgren [2] has reported that ionized hydrogen is limited to certain rather sharply bounded regions in space surroundings, for example 0-type stars and clusters of such stars, and that the gas outside these regions is essentially non-ionized. Other examples of such situations are given by Alfvén's [3] theory of the origin of the planetary system, where a high ionization rate is suggested to appear from collisions between a plasma and a neutral gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds (Piddington [4], Lehnert [5]). Following Hans [6], the medium may be idealized as a mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. Hans [6] and Bhatia [7] have shown that the collisions have a stabilizing effect on the Rayleigh-Taylor instability. The magnetic field has been considered to be uniform in the above studies. Generally the magnetic field has a stabilizing effect on the instability, but there are a few exceptions also. For example, Kent [8] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that the system is unstable under certain conditions, while in the absence of a magnetic field the system is known to be stable. The medium has been considered to be non-porous in all the above studies.

Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vice versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context (McDonnel [9]). In stellar interiors and atmospheres, the magnetic field may be (and quite often is) variable (and nonuniform) and may altogether alter the nature of the instability. Often, the plasma is not fully ionized and may be mixed with neutral atoms, and so collisional effects are important. The present paper deals with the Rayleigh-Taylor instability of a partially ionized plasma in a porous medium in presence of a variable magnetic field perpendicular to gravity. The problem is relevant and important in several situations of geophysics and astrophysics.

2. Perturbation Equations

Here we consider an incompressible layer consisting of an infinitely conducting ionized (hydromag-

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netic) fluid of density ϱ, mixed with neutrals of density ϱ_d, arranged in horizontal strata and acted on by a variable horizontal magnetic field $\boldsymbol{H}(H_0(z), 0, 0)$ and a gravity force $\boldsymbol{g}(0, 0, -g)$. Assume that both the ionized and neutral components behave like continuum fluids and that the effects on the neutral component result from the presence of the magnetic field, gravity and pressure being neglected. This composite plasma layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.

The hydromagnetic equations governing the motion of the composite plasma through porous medium are

$$\frac{\varrho}{\varepsilon}\left[\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{q} \cdot \nabla)\boldsymbol{q}\right] \tag{1}$$

$$= -\nabla p + g\,\varrho + \frac{\mu_e}{4\pi}(\nabla \times \boldsymbol{H}) \times \boldsymbol{H} - \frac{\mu}{k_1}\boldsymbol{q} + \frac{\varrho_d\,v_c}{\varepsilon}(\boldsymbol{q}_d - \boldsymbol{q}),$$

$$\nabla \cdot \boldsymbol{q} = 0, \tag{2}$$

$$\frac{\partial \boldsymbol{q}_d}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{q}_d \cdot \nabla)\boldsymbol{q}_d = -v_c(\boldsymbol{q}_d - \boldsymbol{q}), \tag{3}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{4}$$

$$\varepsilon \frac{\partial \boldsymbol{H}}{\partial t} = (\boldsymbol{H} \cdot \nabla)\boldsymbol{q} - (\boldsymbol{q} \cdot \nabla)\boldsymbol{H}, \tag{5}$$

$$\left(\varepsilon \frac{\partial}{\partial t} + \boldsymbol{q} \cdot \nabla\right)\varrho = 0, \tag{6}$$

where ϱ, p and $\boldsymbol{q}(u, v, w)$ denote respectively the density, pressure and velocity of the hydromagnetic fluid. $\boldsymbol{q}_d$, v_c, μ and μ_e stand for the velocity of the neutral component, the mutual collisional (frictional) frequency between the two components of the composite medium, the viscosity of the hydromagnetic (ionized) fluid and the magnetic permeability, respectively. Equations (1) and (2) represent the equations of motion and continuity for the hydromagnetic fluid whereas (3) is the equation of motion of the neutral component under the assumptions mentioned above. Equations (4) and (5) are the Maxwell ones whereas (6) represents the fact that the density of a particle remains unchanged as we follow it with its motion.

The initial stationary state whose stability we wish to examine is that of an incompressible hydromagnetic (ionized) fluid of variable density and viscosity arranged in horizontal strata permeated with neutral particles in porous medium. The system is acted on by a variable horizontal magnetic field $\boldsymbol{H}(H_0(z), 0, 0)$. Consider an infinite horizontal composite layer of thickness d bounded by the planes $z = 0$ and $z = d$. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then following its further evolution.

Let $\delta\varrho$, δp, $\boldsymbol{q}(u, v, w)$; $\boldsymbol{h}(h_x, h_y, h_z)$ and $\boldsymbol{q}_d$ denote, respectively, the perturbations in the hydromagnetic fluid density $\varrho(z)$, pressure $p(z)$, velocity $(0, 0, 0)$, magnetic field $(H_0(z), 0, 0)$ and neutral component velocity $(0, 0, 0)$. Then the linearized hydromagnetic perturbation equations governing the motion of the composite plasma through porous medium are

$$\frac{\varrho}{\varepsilon}\frac{\partial \boldsymbol{q}}{\partial t} = -\nabla\delta p + g\,\delta\varrho + \frac{\mu_e}{4\pi}[(\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + (\nabla \times \boldsymbol{H}) \times \boldsymbol{h}]$$

$$- \frac{\mu}{k_1}\boldsymbol{q} + \frac{\varrho_d\,v_c}{\varepsilon}(\boldsymbol{q}_d - \boldsymbol{q}), \tag{7}$$

$$\nabla \cdot \boldsymbol{q} = 0, \tag{8}$$

$$\frac{\partial \boldsymbol{q}_d}{\partial t} - = v_c(\boldsymbol{q}_d - \boldsymbol{q}), \tag{9}$$

$$\nabla \cdot \boldsymbol{h} = 0, \tag{10}$$

$$\varepsilon \frac{\partial \boldsymbol{h}}{\partial t} = (\boldsymbol{H} \cdot \nabla)\boldsymbol{q} - (\boldsymbol{q} \cdot \nabla)\boldsymbol{H}, \tag{11}$$

$$\varepsilon \frac{\partial}{\partial t}\delta\varrho = -\omega\left(\frac{d\varrho}{dz}\right). \tag{12}$$

Analyzing the disturbances into normal modes, we assume that the perturbation quantities have an x, y, and t dependence of the form

$$\exp(i k_x x + i k_y y + n t), \tag{13}$$

where k_x and k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$ and n is, in general, a complex constant.

Eliminating $\boldsymbol{q}_d$ between (7) and (9) and using (13), (7)–(12) gives

$$\left(n' + \frac{v}{k_1}\right)\varrho u = -i k_x \delta p + \frac{\mu_e}{4\pi}h_z(DH_0), \tag{14}$$

$$\left(n' + \frac{v}{k_1}\right)\varrho v = -i k_y \delta p + \frac{\mu_e H_0}{4\pi}(i k_x h_y - i k_y h_x), \tag{15}$$

$$\left(n' + \frac{v}{k_1}\right)\varrho w = -D\,\delta p + \frac{\mu_e H_0}{4\pi}$$
$$\cdot\left(i k_x h_z - D h_x - h_x \frac{DH_0}{H_0}\right) - g\,\delta\varrho, \tag{16}$$

$$i k_x u + i k_y v + D w = 0, \tag{17}$$

$$i k_x h_x + i k_y h_y + D h_z = 0, \tag{18}$$

$$\varepsilon n h_x = i k_x H_0 u - w D H_0, \tag{19}$$

$$\varepsilon n h_y = i k_x H_0 v, \tag{20}$$

$$\varepsilon n h_z = i k_x H_0 w, \tag{21}$$

$$\varepsilon n \delta \varrho = - w D \varrho, \tag{22}$$

where

$$n' = \frac{n}{\varepsilon} \left(1 + \frac{\alpha_0 v_c}{n + v_c} \right), \quad \alpha_0 = \frac{\varrho_d}{\varrho}, \quad v = \frac{\mu}{\varrho}$$

and

$$D = \frac{\mathrm{d}}{\mathrm{d}z}.$$

Equation (15), with the help of (19) and (20), becomes

$$\left(n' + \frac{v}{k_1} \right) v = - i k_y \delta p$$
$$+ \frac{\mu_e H_0}{4 \pi \varepsilon n} (i k_x H_0 \zeta + i k_y w D H_0), \tag{23}$$

where $\zeta = i k_x v - i k_y u$ is the z-component of vorticity.

Multiplying (14) by $- i k_x$, (23) by $- i k_y$, adding and using (17), we obtain

$$\left(n' + \frac{v}{k_1} \right) \varrho D w = - k^2 \delta p + \frac{\mu_e k_x k_y H_0^2}{4 \pi \varepsilon n} \zeta$$
$$+ \frac{\mu_e k_y^2 H_0}{4 \pi \varepsilon n} (D H_0) w$$
$$- \frac{i \mu_e k_x}{4 \pi} h_z (D H_0). \tag{24}$$

Eliminating δp between (16) and (24) and using (17)–(22), we get

$$n' [D (\varrho D w) - k^2 \varrho w] + \frac{1}{k_1} [D (\varrho v D w) - k^2 \varrho v w]$$
$$+ \frac{\mu_e H_0^2 k_x^2}{4 \pi n \varepsilon} (D^2 - k^2) w \tag{25}$$
$$+ \frac{\mu_e k_x^2}{4 \pi n \varepsilon} D (H_0^2) D w + \frac{g k^2}{\varepsilon n} (D \varrho) w = 0.$$

3. Two Uniform Partially Ionized Plasmas Separated by a Horizontal Boundary

Consider the case when two superposed partially ionized plasmas of uniform densities ϱ_1 and ϱ_2, uniform viscosities μ_1 and μ_2 and uniform magnetic fields H_{01} and H_{02} are separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 distinguish the lower and the upper fluids, respectively. Then, in each region

of constant ϱ, μ, and H, (25) reduces to

$$(D^2 - k^2) w = 0. \tag{26}$$

The general solution of (26) is

$$w = A e^{+kz} + B e^{-kz}, \tag{27}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied here are:

i) The velocity w should vanish when $z \to - \infty$ (for the lower fluid) and $z \to + \infty$ (for the upper fluid).
ii) $w(z)$ is continuous at $z = 0$.
iii) The pressure should be continuous across the interface. Applying the boundary conditions (i) and (ii), we have

$$w_1 = A e^{+kz}, \quad (z < 0), \tag{28}$$

$$w_2 = A e^{-kz}, \quad (z > 0), \tag{29}$$

the same constant A being chosen to ensure the continuity of w at $z = 0$.

The continuity of pressure implies that

$$n' \Delta_0 (\varrho D w) + \frac{1}{k_1} \Delta_0 (\varrho v D w) + \frac{\mu_e k_x^2}{4 \pi n \varepsilon} \Delta_0 (H_0^2 D w)$$
$$+ \frac{g k^2}{n \varepsilon} \Delta_0 (\varrho) w_0 = 0. \tag{30}$$

Applying the condition (30) to the solutions (28) and (29), we obtain

$$n^3 + [v_c (1 + \alpha_0) + \frac{\varepsilon}{k_1} (\alpha_1 v_1 + \alpha_2 v_2)] n^2$$
$$+ \left[\frac{\varepsilon}{k_1} v_c (\alpha_1 v_1 + \alpha_2 v_2) + 2 k_x^2 V_A^2 - g k (\alpha_2 - \alpha_1) \right] n$$
$$+ v_c [2 k_x^2 V_A^2 - g k (\alpha_2 - \alpha_1)] = 0, \tag{31}$$

where

$$\alpha_{1,2} = \frac{\varrho_{1,2}}{\varrho_1 + \varrho_2}, \quad v_{1,2} = \frac{\mu_{1,2}}{\varrho_{1,2}}.$$

For the sake of simplicity, we assume that the Alfvén velocities of the two fluids are the same, so that

$$V_A^2 = \frac{\mu_e H_{01}^2}{4 \pi (\varrho_1 + \varrho_2)} = \frac{\mu_e H_{02}^2}{4 \pi (\varrho_1 + \varrho_2)}.$$

(a) *Stable Case* $(\varrho_1 > \varrho_2)$

For the potentially stable case, $\alpha_1 > \alpha_2$ and equation (31) does not involve any change of sign and so does not allow any positive root.

The system is therefore stable.

(b) *Unstable Case* $(\varrho_2 > \varrho_1)$

For the potentially unstable case, if

$$2 k_x^2 V_A^2 > g k (\alpha_2 - \alpha_1), \tag{32}$$

(31) does not admit any change of sign and so has no positive root. Therefore the system is stable.
If

$$2 k_x^2 V_A^2 < g k (\alpha_2 - \alpha_1), \tag{33}$$

the constant term in (31) is negative. Equation (31), therefore, allows one change of sign and so has one positive root. The occurrence of a positive root implies that the system is unstable.

Thus for the unstable case $(\varrho_2 > \varrho_1)$, the system is stable or unstable according as $\dfrac{\mu_e H^2 k_x^2}{2\pi}$ is greater than or smaller than $g k (\varrho_2 - \varrho_1)$. In the absence of a magnetic field, (31) has one positive root, and so the system is unstable for $\varrho_2 > \varrho_1$. But the magnetic field has got a stabilizing effect and completely stabilizes the wave-number band $k > k^*$, where

$$k^* = \frac{2\pi g (\varrho_2 - \varrho_1)}{\mu_e H_0^2} \sec^2 \theta, \tag{34}$$

and θ is the inclination of the wave vector $\boldsymbol{k}$ to the direction of the magnetic field $\boldsymbol{H}$ i.e. $k_x = k \cos \theta$.

4. The Case of Exponentially Varying Density, Viscosity, Magnetic Field and Neutral Particles Number Density

Equation (25) can be written as

$$n' [D (\varrho D w) - k^2 \varrho w] + \frac{1}{k_1} [D (\varrho v D w) - k^2 \varrho v w]$$

$$+ \frac{g k^2}{\varepsilon n} (D \varrho) w + \frac{\mu_e k_x^2}{4 \pi \varepsilon n} D (H_0^2 D w)$$

$$- \frac{\mu_e k_x^2 H_0^2 k^2}{4 \pi \varepsilon n} w = 0. \tag{35}$$

Let us assume

$$\varrho = \varrho_0 \, e^{\beta z}, \quad \varrho_d = \varrho_{d_0} \, e^{\beta z},$$
$$\mu = \mu_0 \, e^{\beta z}, \quad H_0^2 (z) = H_1^2 \, e^{\beta z}, \tag{36}$$

where ϱ_0, ϱ_{d_0}, μ_0, H_1, and β are constants. Equations (36) imply that the coefficient of kinematic viscosity v and the Alfvén velocity V_A are constant everywhere.

Substituting the values of ϱ, ϱ_d, μ, H_0^2 in (35) and neglecting the effect of heterogeneity on inertia, we

obtain

$$\left[\frac{n'}{v_0} + \frac{1}{k_1} + \frac{k_x^2 V_A^2}{n v_0 \varepsilon} \right] (D^2 - k^2) \, w + \frac{g \beta k^2}{n v_0 \varepsilon} w = 0, \tag{37}$$

where $v_0 = \mu_0 / \varrho_0$.

Consider the case of two free boundaries. The boundary conditions for the case of two free surfaces are

$$w = D^2 w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{38}$$

The proper solution of (37) satisfying (38) is

$$w = A \sin \frac{m \pi z}{d}, \tag{39}$$

where A is a constant and m is any integer.

Substituting (39) in (37), we obtain the dispersion relation

$$\left[\frac{n'}{v_0} + \frac{1}{k_1} + \frac{k_x^2 V_A^2}{n v_0 \varepsilon} \right] \left[\left(\frac{m \pi}{d} \right)^2 + k^2 \right] - \frac{g \beta k^2}{n v_0 \varepsilon} = 0. \tag{40}$$

Letting $\left(\dfrac{m \pi}{d} \right)^2 + k^2 = L$, the above equation, on simplification, becomes

$$n^3 + \left[v_c (1 + \alpha_0) + \frac{\varepsilon v_0}{k_1} \right] n^2$$

$$+ \left[\frac{v_0 v_c \varepsilon}{k_1} + \left(k_x^2 V_A^2 - \frac{g \beta k^2}{L} \right) \right] n$$

$$+ v_c \left(k_x^2 V_A^2 - \frac{g \beta k^2}{L} \right) = 0. \tag{41}$$

If $\beta < 0$ (stable stratification), (41) does not admit any positive root of n and so the system is always stable for disturbances of all wave numbers.

If $\beta > 0$ (unstable stratification), the system is stable or unstable according as

$$k_x^2 V_A^2 > \text{ or } < \frac{g \beta k^2}{L}. \tag{42}$$

The system is clearly unstable for $\beta > 0$ in the absence of a magnetic field. However, the system can be completely stabilized by a magnetic field as can be seen from (42), if

$$V_A^2 > \frac{g \beta k^2}{k_x^2 L}.$$

The magnetic field therefore succeeds in stabilizing wavenumbers in the range

$$k^2 > \frac{g \beta}{V_A^2} \sec^2 \theta - \left(\frac{m \pi}{d} \right)^2, \tag{43}$$

which were unstable in the absence of a magnetic field. The collisions and medium permeability do not have any qualitative effect on the nature of the stability.

Thus if

$$\beta > 0 \quad \text{and} \quad k_x^2 V_A^2 < \frac{g\beta k^2}{L},$$

(41) has one positive root. Let n_0 denote the positive root of (41). Then

$$n_0^3 + \left[v_c(1+\alpha_0) + \frac{\varepsilon v_0}{k_1} \right] n_0^2 + \left[\frac{v_0 v_c \varepsilon}{k_1} + \left(k_x^2 V_A^2 - \frac{g\beta k^2}{L} \right) \right] n_0 + v_c \left(k_x^2 V_A^2 - \frac{g\beta k^2}{L} \right) = 0. \tag{44}$$

To study the behaviour of growth rates of unstable modes with respect to viscosity, medium permeability and collisions, we examine the natures of $\dfrac{dn_0}{dv_0}$, $\dfrac{dn_0}{dk_1}$ and $\dfrac{dn_0}{dv_c}$ analytically. Equation (44) yields

$$\frac{dn_0}{dv_0} = - \frac{\dfrac{\varepsilon}{k_1} n_0 (n_0 + v_c)}{3 n_0^2 + 2 \left\{ v_c(1+\alpha_0) + \dfrac{\varepsilon v_0}{k_1} \right\} n_0 + \dfrac{v_0 v_c \varepsilon}{k_1} + \left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L} \right)}, \tag{45}$$

$$\frac{dn_0}{dk_1} = \frac{\dfrac{v_0}{k_1^2} n_0 (n_0 + v_c)}{3 n_0^2 + 2 \left\{ v_c(1+\alpha_0) + \dfrac{\varepsilon v_0}{k_1} \right\} n_0 + \dfrac{v_0 v_c \varepsilon}{k_1} + \left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L} \right)}, \tag{46}$$

$$\frac{dn_0}{dv_c} = - \frac{(1+\alpha_0) n_0^2 + \dfrac{\varepsilon v}{k_1} n_0 + \left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L} \right)}{3 n_0^2 + 2 \left\{ v_c(1+\alpha_0) + \dfrac{\varepsilon v_0}{k_1} \right\} n_0 + \dfrac{v_0 v_c \varepsilon}{k_1} + \left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L} \right)}. \tag{47}$$

It is evident from (45) that dn_0/dv_0 may be negative or positive depending on whether the denominator in (45) is positive or negative. The growth rates, therefore, both decrease (for certain wave numbers) and increase (for different wave numbers) with the increase in kinematic viscosity. Similarly, the growth rates are found to increase and decrease with the increase in medium permeability and collisional frequency.

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publication, New York 1981, Chapt. X
[2] B. Strömgren, Astrophys. J. **89**, 526 (1939).
[3] H. Alfvén, The Origin of the Solar System, Clarendon Press, Oxford 1954.
[4] J. H. Piddington, Monthly Notices Roy. Astron. Soc. **14**, 638 (1954).
[5] B. Lehnert, Suppl. Nuovo Cim. **13**, 59 (1959).
[6] H. K. Hans, Nucl. Fusion **8**, 89 (1968).
[7] P. K. Bhatia, Nucl. Fusion **10**, 383 (1970).
[8] A. Kent, Phys. Fluids **9**, 1286 (1966).
[9] J. A. M. McDonnel, Cosmic Dust, John Wiley and Sons, Toronto 1978, p. 330.